Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Pure and Applied Algebra 206 (2006) 66–82
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

On the realization of orbit closures as support varieties ☆

Daniel K. Nakano[a,*], Toshiyuki Tanisaki[b]

[a] Department of Mathematics, University of Georgia, Athens, Georgia 30602, USA
[b] Department of Mathematics, Osaka City University, Osaka, Japan

Received 28 October 2004; received in revised form 14 April 2005
Available online 8 September 2005

Dedicated to Eric M. Friedlander on the occasion of his 60th birthday

Abstract

Let G be a reductive algebraic group over an algebraically closed field k of characteristic $p > 0$ with $\mathfrak{g} = \mathrm{Lie}(G)$. In this paper the authors investigate the following problem. Given a nilpotent orbit $\mathcal{O}$ in the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$, construct a finite-dimensional (tilting) G-module such that the support variety of M, $\mathcal{V}_{\mathfrak{g}}(M)$, is the closure of $\mathcal{O}$.

MSC: Primary: 17B55; 20G; secondary: 17B50

1. Introduction

1.1.

In the early 1980s, Carlson [4] used the spectrum of the cohomology ring to define varieties associated to modules over group algebras. These varieties are often called support varieties. Carlson showed that given a closed conical subvariety Z inside the support variety

☆ Research of the first author (Daniel K. Nakano) partially supported by NSF Grant DMS-0400548. Daniel K. Nakano acknowledges the support of the Clay Mathematical Institute (CMI) who provided his funding for this conference as a CMI Mathematical Emissary.

* Corresponding author.
E-mail addresses: nakano@math.uga.edu (D.K. Nakano), tanisaki@sci.osaka-cu.ac.jp (T. Tanisaki).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2005.04.014

of the trivial module, there exists a finite-dimensional module for the group algebra whose support variety is precisely Z.

Friedlander and Parshall [12], in the mid 1980s, subsequently extended the theory of support varieties to finite-dimensional restricted Lie algebras over algebraically closed fields k of characteristic $p>0$. Let $\mathscr{V}_{\mathfrak{g}}(M)$ denote the support variety of a finite-dimensional module M over a finite-dimensional restricted Lie algebra $\mathfrak{g}$. By definition $\mathscr{V}_{\mathfrak{g}}(M)$ is a closed subvariety of the support variety $\mathscr{V}_{\mathfrak{g}}(k)$ of the trivial $\mathfrak{g}$-module k. Their work along with results of Jantzen [19] and Suslin, Friedlander and Bendel [34] demonstrated that $\mathscr{V}_{\mathfrak{g}}(k)$ can be identified with the restricted nullcone $\mathscr{N}_1(\mathfrak{g})=\{x\in\mathfrak{g}\mid x^{[p]}=0\}$. At the end of their paper, Friedlander and Parshall [12, Section 3] posed the following questions.

(1.1.1) For which finite-dimensional restricted Lie algebras $\mathfrak{g}$ over k, is $\mathscr{V}_{\mathfrak{g}}(k)$ irreducible?

(1.1.2) Let G be a connected reductive algebraic group over k with $\mathfrak{g}=\mathrm{Lie}(G)$. Given a G-stable closed conical subvariety Z in $\mathscr{V}_{\mathfrak{g}}(k)$, is there a finite-dimensional G-module M with $\mathscr{V}_{\mathfrak{g}}(M)=Z$?

Assume that $\mathfrak{g}=\mathrm{Lie}(G)$ for a connected reductive algebraic group G. Then $\mathscr{N}_1(\mathfrak{g})$ is a G-invariant closed subvariety inside the nullcone $\mathscr{N}(\mathfrak{g})$ of $\mathfrak{g}$. If $p\geqslant h$ where h is the Coxeter number of G, then we have $\mathscr{N}_1(\mathfrak{g})=\mathscr{N}(\mathfrak{g})$, and hence (1.1.1) is true in this case. Moreover, Parshall, Vella and the first author [30] have recently proved that $\mathscr{N}_1(\mathfrak{g})$ is indeed irreducible when p is good. In fact, $\mathscr{N}_1(\mathfrak{g})$ is the closure of some Richardson orbit. Carlson, Lin, Parshall and the first author [7] provided an explicit description of the Richardson orbit determining the restricted nullcone.

The aim of this paper is to investigate the question (1.1.2). Using the same proof given by Carlson, one can easily show there is a restricted $\mathfrak{g}$-module whose support is Z. However, not every module for the Lie algebra $\mathfrak{g}$ lifts to a module for the algebraic group G. Pevtsova [31, Corollary 3.11] has shown that (1.1.2) has an affirmative answer when the assumption on finite-dimensionality is dropped.

Since there are finitely many G-orbits on $\mathscr{N}_1(\mathfrak{g})$, question (1.1.2) is equivalent to the following question.

(1.1.3) Given an orbit $\mathscr{O}$ in $\mathscr{N}_1(\mathfrak{g})$, is there a finite-dimensional G-module whose support variety is $\overline{\mathscr{O}}$?

Jantzen [20, 2.8] first demonstrated that (1.1.3) has an affirmative answer when the root system $\Phi=A_n$, and $\Phi=B_2$. During a recent workshop at the Korea Institute for Advanced Study on modular representation theory, Parshall proposed a stronger version of (1.1.3).

(1.1.4) Given an orbit $\mathscr{O}$ in $\mathscr{N}_1(\mathfrak{g})$, is there a finite-dimensional G-module with a good filtration whose support variety is $\overline{\mathscr{O}}$?

The question (1.1.4) is equivalent to the statement when good filtration is replaced by Weyl filtration because the support varieties of M and its contragredient dual module M^* are equal. By [7, Theorem 4.8] (1.1.4) has a positive answer when p is good and Φ is any irreducible root system (i.e. A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2) for any Richardson orbit inside $\mathscr{N}_1(\mathfrak{g})$. In this situation each Richardson orbit can be realized as the support variety of an induced module. One can even formulate a stronger version of (1.1.4).

(1.1.5) Given an orbit $\mathscr{O}$ in $\mathscr{N}_1(\mathfrak{g})$, is there a tilting G-module whose support variety is $\overline{\mathscr{O}}$?

The main results of this paper involve constructing tilting modules whose support varieties realize orbit closures. Our results indicate that the tilting modules will be the appropriate class of G-modules to represent the entire collection of orbit closures as support varieties. Furthermore, our findings are consistent in spirit with Humphreys' conjecture on the support varieties of tilting modules for $p \geqslant h$ [18, Section 12]. For a proof of this conjecture in the quantum group case we refer the reader to [3]. However, it should be pointed out that our results differ from the point of view of the conjecture because we are not just considering tilting modules corresponding to regular dominant integral weights. In [30], it was shown that the closures of Richardson classes in the restricted nullcone can be represented by using Weyl modules. This correspondence led to a new representation theoretic proof of the Johnston–Richardson theorem. Indeed, a correspondence between tilting modules and general orbit closures would be highly desirable because it may lead to some new and interesting insights for both sets of objects.

1.2.

The paper is organized as follows. Throughout this paper we will always assume that the underlying characteristic of the field is good. In Section 2 the basic concepts for this paper are outlined. At the end of the section, (1.1.5) is verified for all Richardson orbits contained in $\mathcal{N}_1(\mathfrak{g})$ via simple tilting modules. In Section 3, we provide an affirmative answer to (1.1.5) for classical simple Lie algebras (where the root system Φ is of type A_n, B_n, C_n or D_n). In Section 4, we investigate (1.1.2)–(1.1.4) for the exceptional simple Lie algebras. It is proved that for Φ of type G_2 and F_4 (1.1.3) has an affirmative answer. For the other exceptional algebra $\Phi = E_6$, E_7, E_8, we provide an affirmative answer to (1.1.3) for all but a small number of non-Richardson orbits. Improvements to these results (i.e. affirmative answers to (1.1.5)) for the exceptional Lie algebras E_6, E_7, E_8, F_4, G_2 can be made if the prime is sufficiently large.

2. Preliminaries

2.1.

Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathfrak{g}$ be an arbitrary finite-dimensional restricted Lie algebra over k and let $u(\mathfrak{g})$ be its restricted enveloping algebra. It is well-known that $u(\mathfrak{g})$ is a finite-dimensional cocommutative Hopf algebra. Set

$$\mathrm{H}(u(\mathfrak{g}), k) = \begin{cases} \mathrm{H}^{2\bullet}(u(\mathfrak{g}), k) & \text{if char } k \neq 2, \\ \mathrm{H}^{\bullet}(u(\mathfrak{g}), k) & \text{if char } k = 2. \end{cases}$$

The cohomology ring $R = \mathrm{H}(u(\mathfrak{g}), k)$ is a commutative, finitely generated k-algebra [14,12]. Given a finite-dimensional $u(\mathfrak{g})$-module M, define the *support variety* $\mathcal{V}_{\mathfrak{g}}(M)$ as follows. Let $J := J(M)$ be the annihilator ideal in R for its action on $\mathrm{Ext}^{\bullet}_{u(\mathfrak{g})}(M, M)$. The *support variety* $\mathcal{V}_{\mathfrak{g}}(M)$ is defined as the maximal ideal spectrum of R/J. Support varieties are compatible with taking direct sums and tensor products of modules. For finite-dimensional $u(\mathfrak{g})$-modules, M and N, one has the following properties [12, Proposition 2.1(b)(c)]:

(2.1.1) $\mathscr{V}_{\mathfrak{g}}(M \oplus N) = \mathscr{V}_{\mathfrak{g}}(M) \cup \mathscr{V}_{\mathfrak{g}}(N)$
(2.1.2) $\mathscr{V}_{\mathfrak{g}}(M \otimes N) = \mathscr{V}_{\mathfrak{g}}(M) \cap \mathscr{V}_{\mathfrak{g}}(N)$.

Let G be an algebraic group over k and $\mathfrak{g} = \mathrm{Lie}(G)$. The Lie algebra $\mathfrak{g}$ is a restricted Lie algebra with p-mapping $x \to x^{[p]}$. According to [34, (1.6), (5.11)],

(2.1.3) $\mathscr{V}_{\mathfrak{g}}(k) \cong \mathscr{N}_1(\mathfrak{g}) := \{x \in \mathfrak{g} : x^{[p]} = 0\ \}$.

Furthermore, under this identification, if M is a finite-dimensional $u(\mathfrak{g})$-module then

(2.1.4) $\mathscr{V}_{\mathfrak{g}}(M) \cong \{x \in \mathfrak{g} : x^{[p]} = 0,\ \ M$ is not free as $u(\langle x \rangle)$-module$\} \cup \{0\}$.

The group G acts on $\mathscr{N}_1(\mathfrak{g})$ by conjugation and if M is a G-module, then $\mathscr{V}_{\mathfrak{g}}(M)$ is a G-stable subvariety of $\mathscr{N}_1(\mathfrak{g})$. Let G be a closed subgroup of an algebraic group K with $\mathrm{Lie}(K) = \mathfrak{k}$. If M is a finite-dimensional K-module M one has by [12, Proposition 2.1a]

(2.1.5) $\mathscr{V}_{\mathfrak{g}}(M) = \mathscr{V}_{\mathfrak{k}}(M) \cap \mathfrak{g}$

which shows that support varieties behave naturally with respect to inclusions.

2.2.

In the rest of this paper, let G be a simple algebraic group defined over k. Let T be a maximal torus of G. We denote its character group by $X(T)$ and set $X^*(T) = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$. The canonical pairing $X(T) \times X^*(T) \to \mathbb{Z}$ is denoted by $(\lambda, \chi) \mapsto \langle \lambda, \chi \rangle$. The root system Φ with respect to (G, T) is identified with a subset of $X(T)$. We fix a set Φ^+ of positive roots and set $\Phi^- = -\Phi^+$. The set of simple roots determined by Φ^+ is denoted by $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. We will use the same ordering of simple roots given in [21] following the standard conventions in Bourbaki. Let W be the Weyl group and the W_p be the affine Weyl group associated to Φ.

For $\alpha \in \Phi$ we denote the corresponding coroot by $\alpha^\vee \in X^*(T)$. The Coxeter number h is defined by $h = \langle \rho, \alpha_0^\vee \rangle + 1$, where ρ is the half-sum of positive roots and α_0 is the highest short root. For $1 \leqslant i \leqslant \ell$, let ω_i be the fundamental dominant weight satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The dominant weights $X(T)_+$ consist of those $\lambda \in X(T)$ with $\langle \lambda, \alpha_i^\vee \rangle \geqslant 0$, $1 \leqslant i \leqslant \ell$. Let $B \supset T$ be the Borel subgroup defined by the negative roots $-\Phi^+$. If $J \subset \Delta$ then $P_J = L_J \ltimes U_J$ is the parabolic subgroup determined by J with Levi factor L_J. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{p}_J$, $\mathfrak{u}_J$ be the Lie algebras of G, B, P_J, U_J.

Throughout this paper we will always assume that p is a good prime for Φ. A prime is *good* if and only if the prime does not appear as a coefficient in the decomposition of a root into simple roots. A list of good primes is provided below.

- Φ of type A_n, all primes.
- Φ of type B_n, C_n, D_n, $p \geqslant 3$.
- Φ of type E_6, E_7, F_4, G_2, $p \geqslant 5$.
- Φ of type E_8, $p \geqslant 7$.

Let $\mathscr{N}(\mathfrak{g})$ be the variety of nilpotent elements of $\mathfrak{g}$. The variety $\mathscr{N}(\mathfrak{g})$ is often called the nullcone. The nullcone is an irreducible variety of dimension equal to $|\Phi|$. The group G acts on $\mathscr{N}(\mathfrak{g})$ via conjugation and $\mathscr{N}(\mathfrak{g})$ has finitely many G-orbits. For good primes the classification and structures of these orbits coincide precisely with the orbit theory for complex simple Lie algebras (see [8,9,11,17]). We also note that the closure relations on orbits are given in [15] for classical groups, [28,29] for E_6, E_7, E_8, and [33] for F_4. If $J \subseteq \Delta$

then $G \cdot \mathfrak{u}_J$ is a closed, irreducible subvariety of $\mathcal{N}(\mathfrak{g})$ of dimension equal to $2\dim \mathfrak{u}_J$. There exists a unique dense open G-orbit in $G \cdot \mathfrak{u}_J$. Orbits which arise in this way are the Richardson orbits in $\mathfrak{g}$.

For a given G-module M, let M^* be the contragredient dual of M and let M^τ denote the transposed dual of M as defined in [21, II 2.12]. For $\lambda \in X(T)_+$, let $H^0(\lambda)$ be the induced module $\mathrm{ind}_B^G(\lambda)$. Also, let $V(\lambda)$ be the corresponding Weyl module, and $L(\lambda)$ be the simple G-module with highest weight λ. One has $V(\lambda) \cong \mathrm{ind}_B^G(\lambda)^\tau$. Moreover, $H^0(\lambda)$ and $V(\lambda)$ have the same characters described by Weyl's character formula. A filtration of a G-module is called good if each subquotient relative to the filtration is isomorphic to $H^0(\lambda)$ for some $\lambda \in X(T)_+$. A G-module M is called tilting if both M and M^τ have good filtrations (see [21, II.E]). For $\lambda \in X(T)_+$ there exists a unique indecomposable tilting module $T(\lambda)$ with highest weight λ. Any tilting G-module is a direct sum of the indecomposable tilting modules. Finally, if $H^0(\lambda)$ is a simple G-module, then $H^0(\lambda) \cong V(\lambda) \cong L(\lambda) = T(\lambda)$.

2.3.

From our discussion in Section 2.1, $\mathcal{N}_1(\mathfrak{g})$ is a G-stable subvariety of the nullcone $\mathcal{N}(\mathfrak{g})$. The variety $\mathcal{N}_1(\mathfrak{g})$ in this context is called the restricted nullcone. Since the restricted nullcone is a G-invariant closed subvariety of $\mathcal{N}(\mathfrak{g})$, it follows that if M is a G-module then $\mathcal{V}_{\mathfrak{g}}(M)$ is a finite union of closures of orbits of which there are finitely many possibilities.

For $\lambda \in X(T)$, consider the set

$$\Phi_\lambda = \{\alpha \in \Phi : \langle \lambda + \rho, \alpha^\vee \rangle \in p\mathbb{Z}\}.$$

Since p is good there exists a $w \in W$ such that $w(\Phi_\lambda) = \Phi_J$ for some $J \subseteq \Delta$. Note that $w(\Phi_\lambda) = \Phi_{w\cdot\lambda}$ and $\Phi_\lambda = \Phi_{\lambda+p\nu}$ for all $\lambda, \nu \in X(T)$ and $w \in W$. The following result [30, (6.2.1) Theorem] provides a description of the support varieties of the modules $H^0(\lambda)$ for $\lambda \in X(T)_+$ in terms of closures of Richardson orbits.

Theorem 2.3. *Let $\lambda \in X(T)_+$ and $w \in W$ such that $w(\Phi_\lambda) = \Phi_J$ for some $J \subseteq \Delta$. Then $\mathcal{V}_{\mathfrak{g}}(H^0(\lambda)) = G \cdot \mathfrak{u}_J$.*

The preceding result shows that $\mathcal{N}_1(\mathfrak{g}) = \mathcal{V}_{\mathfrak{g}}(k) = G \cdot \mathfrak{u}_J$ for some $J \subseteq \Delta$. This demonstrates that $\mathcal{N}_1(\mathfrak{g})$ is an irreducible variety whose dimension is equal to $\dim G \cdot \mathfrak{u}_J = |\Phi| - |\Phi_0|$. An explicit description of $\mathcal{N}_1(\mathfrak{g})$ was recently provided in [7, Section 4.4].

In [7], it was shown that for Φ an irreducible root system, restricted parabolics are equivalent to strongly restricted parabolics. From this statement, one can deduce that if $G \cdot \mathfrak{u}_J \subseteq \mathcal{N}_1(\mathfrak{g})$ then there exists $\lambda \in X(T)_+$ such that $G \cdot \mathfrak{u}_J = \mathcal{V}_{\mathfrak{g}}(H^0(\lambda))$.

2.4.

Richardson orbits: Let $\uparrow$ be the Strong Linkage relation on $X(T)$ as defined in [21, II 6.4]: $\lambda \uparrow \mu$ if and only if there exist $\mu_1, \mu_2, \ldots, \mu_t \in X(T)$ and reflections $s_1, s_2, \ldots, s_{t+1} \in W_p$

such that

$$\lambda \leqslant s_1 \cdot \lambda = \mu_1 \leqslant s_2 \cdot \mu_1 = \mu_2 \leqslant \cdots \leqslant s_t \cdot \mu_{t-1} = \mu_t \leqslant s_{t+1} \cdot \mu_t = \mu,$$

where $\cdot$ denotes the "dot action" of the affine Weyl group. We will now show that Theorem 2.3 can be used to realize the closures of Richardson orbits via tilting modules.

Theorem 2.4. *Let G be a simple group with p good. If $\mathcal{O}$ is a Richardson orbit in $\mathcal{N}_1(\mathfrak{g})$ then there exists a simple tilting G-module M such that $\mathcal{V}_{\mathfrak{g}}(M) = \overline{\mathcal{O}}$.*

Proof. Let $\mathcal{O}$ be a Richardson orbit in $\mathcal{N}_1(\mathfrak{g})$. According to [7, Theorem 4.8] there exists $\lambda \in X(T)_+$ such that $\mathcal{V}_{\mathfrak{g}}(H^0(\lambda)) = \overline{\mathcal{O}}$. Let $Y = \{\mu \in X(T)_+ : \mu \uparrow \lambda\}$. Then Y is a non-empty finite set of dominant weights and must have a minimal element with respect to $\uparrow$. Let σ be a minimal element in Y. It follows by [21, II 6.16 Proposition] that $M := H^0(\sigma)$ is a simple module and $M = T(\sigma)$. Furthermore,

$$\overline{\mathcal{O}} = \mathcal{V}_{\mathfrak{g}}(H^0(\lambda)) = \mathcal{V}_{\mathfrak{g}}(H^0(\sigma)) = \mathcal{V}_{\mathfrak{g}}(M)$$

by the fact that σ and λ are linked under the action of W_p and Theorem 2.3. $\square$

We should note that not every orbit closure can be realized as the support variety of a simple G-module. The result above shows that for Φ of type A_n, every orbit closure in $\mathcal{N}_1(\mathfrak{g})$ is the support variety of some simple G-module because all orbits in this case are Richardson. For Φ of type B_2 and G_2, it was shown in [30, (6.6.1) Corollary] that the closure of the minimal orbit cannot be realized as the support variety of a simple G-module.

3. Classical groups

3.1.

Let G be a simple algebraic group and $\varphi : G \to \mathrm{SL}(V)$ be a faithful representation of G. Our first result provides sufficient conditions for when an induced module $H^0(\lambda)$ for $\mathrm{SL}(V)$ admits a good filtration upon restriction to G.

Proposition 3.1. *Let G be a simple algebraic group and $\varphi : G \hookrightarrow K := \mathrm{SL}_N(k)$ be a faithful representation of G. Let $\omega_1, \omega_2, \ldots, \omega_{N-1}$ be the fundamental weights for the root system corresponding to K. For a dominant weight λ for K we denote the corresponding induced module for K by $H^0_K(\lambda)$. We assume that $H^0_K(\omega_i)|_G$ admits a good G-filtration for each $i = 1, 2, \ldots, N-1$.*

(a) *If a K-module M admits a good K-filtration, then $M|_G$ also admits a good G-filtration.*

(b) *If M is a tilting K-module, then $M|_G$ is a tilting G-module.*

Proof. By using the τ-duality part (b) follows from part (a). We proceed to prove part (a). Without a loss of generality one may reduce to the case when $M := H^0_K(\lambda)$ for a dominant weight $\lambda = n_1\omega_1 + n_2\omega_2 + \cdots n_{N-1}\omega_{N-1}$. Set $N = H^0_K(\omega_1)^{\otimes n_1} \otimes H^0_K(\omega_2)^{\otimes n_2} \otimes \cdots \otimes H^0_K(\omega_{N-1})^{\otimes n_{N-1}}$. Then we have an exact sequence

$$0 \to L \to N \to M \to 0$$

of K-modules, where L denotes the kernel of the canonical K-homomorphism $N \to M$ ([21, II 14.20 Proposition]).

We first show that L admits a good K-filtration. This is equivalent to $\mathrm{Ext}^1_K(V_K(\sigma), L) = 0$ for any dominant weight σ, where $V_K(\sigma)$ denotes the Weyl module for K with highest weight σ. The module N has a good K-filtration because it is a tensor product of induced modules [25]. Therefore, we have an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_K(V_K(\sigma), L) \to \mathrm{Hom}_K(V_K(\sigma), N) \to \mathrm{Hom}_K(V_K(\sigma), M) \\ &\to \mathrm{Ext}^1_K(V_K(\sigma), L) \to 0. \end{aligned}$$

If $\sigma \neq \lambda$, we have $\mathrm{Hom}_K(V_K(\sigma), M) = 0$ [21, II 4.13 Proposition] and hence $\mathrm{Ext}^1_K(V_K(\sigma), L) = 0$. If $\sigma = \lambda$ then $\mathrm{Hom}_K(V_K(\lambda), L) = 0$ because the weights of L are less than λ and $\mathrm{Hom}_K(V_K(\lambda), M) \cong k$ [21, II 4.13 Proposition]. Now by taking the transposed dual τ, one has

$$N^\tau \cong V_K(\omega_1)^{\otimes n_1} \otimes V_K(\omega_2)^{\otimes n_2} \otimes \cdots \otimes V_K(\omega_{N-1})^{\otimes n_{N-1}}.$$

It follows that by using this isomorphism and Frobenius reciprocity

$$\mathrm{Hom}_K(V_K(\lambda), N) \cong \mathrm{Hom}_K(N^\tau, H^0_K(\lambda)) \cong \mathrm{Hom}_B(N^\tau, \lambda) \cong k.$$

Here B is the Borel subgroup of K. The last isomorphism uses the fact that the dim $N^\tau_\lambda = 1$ and all other weights are strictly less than λ. Consequently, $\mathrm{Ext}^1_K(V_K(\lambda), L) = 0$ and L has a good K-filtration.

Now let us show that $M|_G = H^0_K(\lambda)|_G$ has a good G-filtration by induction on λ. We have to show $\mathrm{Ext}^1_G(V(\xi), M|_G) = 0$ for any dominant weight ξ for G. From the long exact sequence one has

$$\to \mathrm{Ext}^1_G(V(\xi), N|_G) \to \mathrm{Ext}^1_G(V(\xi), M|_G) \to \mathrm{Ext}^2_G(V(\xi), L|_G) \to .$$

It suffices to show that $\mathrm{Ext}^1_G(V(\xi), N|_G) = \mathrm{Ext}^2_G(V(\xi), L|_G) = 0$. Hence it is enough to show that both $N|_G$ and $L|_G$ admit good G-filtrations. The assertion for $N|_G$ follows from our assumption, and the one for $L|_G$ is a consequence of the hypothesis of induction since L admits a good K-filtration and because all the weights of L are less than λ. $\square$

3.2.

Set $N := N(l) = 2l + 1$ (resp. $2l$) for Φ or type B_l (resp. C_l or D_l). Consider the groups $G = \mathrm{SO}_N(k)$ (resp. $\mathrm{Sp}_N(k)$, $\mathrm{SO}_N(k)$). There exists an embedding via the standard representation of G into $\mathrm{SL}_N(k)$.

Theorem 3.2. *Let G be as above and let $\varphi : G \hookrightarrow K := \mathrm{SL}_N(k)$ be the embedding via the standard representation.*

(a) *If a K-module M admits a good filtration, then* $M|_G$ *also admits a good filtration.*
(b) *If M is a tilting K-module, then* $M|_G$ *is a tilting G-module.*

Proof. From Proposition 3.1, it suffices to show that $H^0_K(\omega_i)|_G$ has a good G-filtration for each $i = 1, 2, \ldots, N-1$. Now $H^0_K(\omega_i) = \Lambda^i(V)$ as a K-module for $i = 1, \ldots, N-1$ where Λ^i denotes the ith exterior power, and $V \cong k^N$. Furthermore,

$$H^0_K(\omega_{N-i}) \cong L_K(\omega_{N-i}) \cong L_K(\omega_i)^* \cong H^0_K(\omega_i)^*$$

for $i = 1, \ldots, l$ [21, II 2.13(1), 2.15]. Therefore, in order to prove the theorem we need to show that $H^0_K(\omega_i)|_G$ is a tilting module for $i = 1, 2, \ldots, l$.

For $\Phi = B_l$ and D_l, $p \neq 2$, $H^0_K(\omega_i)|_G$ is a simple G-module for $i = 1, 2, \ldots, l$ [1, p. 509] and is isomorphic to an induced module for G. It follows that $H^0_K(\omega_i)|_G$ is a tilting module. The characters of these modules are described in [10, 4.1]. On the other hand, for $\Phi = C_l$, $H^0_K(\omega_i)|_G$ is in general not simple, but is still a tilting module. The decomposition into indecomposable tilting modules is provided in [26, Proposition 6.3.1]. □

For $\Phi = C_l$ the structure of $H^0_K(\omega_i)|_G$ $i = 1, 2, \ldots, N-1$ can be quite complicated. This was first observed in work of Premet and Suprunenko [32].

3.3.

For $X = A$, (resp. B, C, D), let $\mathscr{P}_X(N)$ be the set of partitions of N parametrizing the set of nilpotent orbits for A_{N-1} (resp. B_l, C_l, D_l). We give a precise description of $\mathscr{P}_X(N)$ [9, Theorems 5.1.2–5.1.4].

- $\mathscr{P}_A(N)$: all partitions of N.
- $\mathscr{P}_B(N)$: partitions of N such that even parts occur with even multiplicity.
- $\mathscr{P}_C(N)$: partitions of N such that odd parts occur with even multiplicity.
- $\mathscr{P}_D(N)$: partitions of N such that even parts occur with even multiplicity.

A very even partition in $\mathscr{P}_D(N)$ is a partition of N with only even parts. If $\lambda \in \mathscr{P}_X(N)$, let $\mathscr{O}_\lambda$ be the corresponding nilpotent X-orbit. In the case for type D, for very even partitions λ, there are two orbits corresponding to the partition λ. We will denote the two orbits by $\mathscr{O}^I_\lambda$ and $\mathscr{O}^{II}_\lambda$. Let $\mathscr{P}_X(N)_{\mathrm{res}}$ be the set of partitions $\lambda \in \mathscr{P}_X(N)$ such that $\mathscr{O}_\lambda \subseteq N_1(\mathfrak{g})$.

3.4.

We can now provide an affirmative answer to (1.1.5) for G simple and Φ of type A_n, B_n, C_n or D_n.

Theorem 3.4. *Let G be a classical semisimple group where* $\Phi = A_l, B_l, C_l$ *or* D_l *with p good. If* $\mathscr{O}$ *is an orbit in* $\mathscr{N}_1(\mathfrak{g})$ *then there exists a tilting G-module M such that* $\mathscr{V}_{\mathfrak{g}}(M) = \overline{\mathscr{O}}$.

Proof. For $\Phi = A_l$ all orbits are Richardson so the result follows from Theorem 2.4. Let $\Phi = B_l, C_l, D_l$. For any $\lambda \in \mathscr{P}_X(N)_{\mathrm{res}}$ $(X = B, C, D)$ where λ is not very even for $X = D$

$$\mathscr{O}_\lambda = \mathfrak{g} \cap \mathscr{O}^{\mathfrak{sl}_N}_\lambda, \quad \overline{\mathscr{O}}_\lambda = \mathfrak{g} \cap \overline{\mathscr{O}}^{\mathfrak{sl}_N}_\lambda,$$

where $\mathcal{O}_\lambda^{\mathfrak{sl}_N}$ is the corresponding orbit for the partition λ in $\mathfrak{sl}_N(k)$. Since $\lambda \in \mathscr{P}_X(N)_{\text{res}}$, it follows that $\mathcal{O}_\lambda^{\mathfrak{sl}_N} \subseteq \mathcal{N}_1(\mathfrak{sl}_N(k))$. We have already shown that for Φ of type A_{N-1} there exists a tilting $\mathrm{SL}_N(k)$-module M such that $\mathcal{V}_{\mathfrak{sl}_N(k)}(M) = \overline{\mathcal{O}}_\lambda^{\mathfrak{sl}_N}$. Therefore, by (2.1.5)

$$\overline{\mathcal{O}}_\lambda = \mathfrak{g} \cap \overline{\mathcal{O}}_\lambda^{\mathfrak{sl}_N} = \mathfrak{g} \cap \mathcal{V}_{\mathfrak{sl}_N(k)}(M) = \mathcal{V}_{\mathfrak{g}}(M|_G).$$

Furthermore, $M|_G$ is a tilting module by Theorem 3.2(b).

Now consider the case when $\lambda \in \mathscr{P}_D(N)_{\text{res}}$ where λ is very even (note that we have $N = 4m$ for some positive integer m by the existence of a very even partition). Then

$$\mathcal{O}_\lambda^I \cup \mathcal{O}_\lambda^{II} = \mathfrak{g} \cap \mathcal{O}_\lambda^{\mathfrak{sl}_N}.$$

By looking at the elementary divisors given by the procedure in [8, p. 395] one can see that the orbits $\mathcal{O}_\lambda^I$ and $\mathcal{O}_\lambda^{II}$ are even orbits and hence Richardson. For these orbits the result now follows by Theorem 2.4. □

4. Exceptional groups

4.1.

Let G be a simple algebraic group with char $k = p$ good. Moreover, let $\Psi : G \to K := \mathrm{SL}(V)$ be a faithful finite-dimensional representation of G. Set $\mathfrak{g} = \mathrm{Lie}(G)$, $N = \dim V$ and let $\trianglelefteq$ denote the dominance ordering on partitions of N. For λ a partition of N, let $\mathcal{O}^\lambda$ be the K-orbit in $\mathcal{N}(\mathfrak{sl}_N(k))$ corresponding to the matrix with Jordan blocks of size λ. Note that we have $\overline{\mathcal{O}}^\mu \subseteq \overline{\mathcal{O}}^\lambda$ if and only if $\mu \trianglelefteq \lambda$. Now let $\preccurlyeq$ denote the ordering in $\mathcal{N}(\mathfrak{g})$ given by the inclusion of the closures of orbits. If $e_1, e_2 \in \mathcal{N}(\mathfrak{g})$ then $e_1 \preccurlyeq e_2$ if and only if $\overline{G \cdot e_1} \subseteq \overline{G \cdot e_2}$. Furthermore, we say that $e_1 \trianglelefteq e_2$ if and only if the partition corresponding to the Jordan blocks of $\Psi(e_1)$ is less than or equal to the partition corresponding to $\Psi(e_2)$ in the dominance ordering.

Throughout this discussion, let Γ be a complete set of G-orbit representatives in $\mathcal{N}(\mathfrak{g})$. We say that (Γ, Ψ) is *orbit faithful* if and only if for $e_1, e_2 \in \Gamma$, $\mathrm{SL}(V) \cdot e_1 = \mathrm{SL}(V) \cdot e_2$ implies that $G \cdot e_1 = G \cdot e_2$. The following lemma compares the G-orbits in $\mathcal{N}(\mathfrak{g})$ with K-orbits in $\mathcal{N}(\mathfrak{sl}(V))$.

Lemma 4.1. *If* (Γ, Ψ) *is orbit faithful then* $\mathrm{SL}(V) \cdot e \cap \mathfrak{g} = G \cdot e$ *for all* $e \in \Gamma$.

Proof. Observe that $\mathrm{SL}(V) \cdot e \cap \mathfrak{g} = \bigcup_{i=1}^{s} G \cdot y_i$ with $y_i \in \Gamma$ for all i. For $i = 1, 2, \ldots, s$, $y_i \in \mathfrak{g}$ and $y_i \in \mathrm{SL}(V) \cdot e$. Therefore, $\mathrm{SL}(V) \cdot y_i = \mathrm{SL}(V) \cdot e$. Hence, by our assumption $G \cdot y_i = G \cdot e$. □

Later we will see that for simple algebraic groups in good characteristics, a minimal dimensional representation ρ admits a complete set of nilpotent orbit representatives Γ for $\mathcal{N}(\mathfrak{g})$ where (Γ, ρ) is orbit faithful.

4.2.

We have to consider what happens when one intersects closures of orbits in $\mathcal{N}(\mathfrak{sl}(V))$ with $\mathfrak{g}$. Assume that (Γ, Ψ) is orbit faithful then Γ can be identified with a subset of $\mathcal{P}_A(N)$. For $e \in \Gamma$, set

$$\Gamma_e^{\max} = \{f \in \Gamma : f \trianglelefteq e, \text{ and } f \text{ is a maximal element with respect to } \preccurlyeq\}.$$

Proposition 4.2. *Assume that* (Γ, Ψ) *is orbit faithful. Let* $e, f \in \Gamma$.

(a) *If* $f \trianglelefteq e$ *implies that* $f \preccurlyeq e$ *then*

$$\overline{\mathrm{SL}(V) \cdot e} \cap \mathfrak{g} = \overline{G \cdot e}.$$

(b) *More generally, one has*

$$\overline{\mathrm{SL}(V) \cdot e} \cap \mathfrak{g} = \bigcup_{f \in \Gamma_e^{\max}} \overline{G \cdot f}.$$

Proof. Let $e \in \Gamma$ and assume that (Γ, Ψ) is orbit faithful. Since $\overline{G \cdot z} = \bigcup_{f \preccurlyeq z} G \cdot f$ and $\overline{\mathrm{SL}(V) \cdot e} \cap \mathfrak{g}$ is closed in $\mathfrak{g}$, it follows that

$$\begin{aligned} \overline{\mathrm{SL}(V) \cdot e} \cap \mathfrak{g} &= \left(\bigcup_{f \trianglelefteq e} \mathrm{SL}(V) \cdot f \right) \cap \mathfrak{g} = \bigcup_{f \trianglelefteq e} \mathrm{SL}(V) \cdot f \cap \mathfrak{g} \\ &= \bigcup_{f \trianglelefteq e} G \cdot f = \bigcup_{f \in \Gamma_e^{\max}} \overline{G \cdot f}. \qquad \square \end{aligned}$$

4.3.

For G a simple group let $\mathcal{U}(G)$ be the set of unipotent elements in G and $\mathcal{U}_1(G) = \{u \in \mathcal{U}(G) : u^p = 1\}$. For good primes, there is a bijection between nilpotent orbits in $\mathfrak{g}$ and unipotent classes in G. Let $\mathrm{Ad} : G \to \mathrm{SL}(\mathfrak{g})$ be the adjoint representation and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{sl}(\mathfrak{g})$ be the representation obtained by differentiating Ad. McNinch [27, Theorem 10] has shown for very good primes that if $u \in \mathcal{U}_1(G)$ and x is a corresponding class in $\mathcal{N}_1(\mathfrak{g})$ then the partitions for the $\mathrm{SL}(\mathfrak{g})$-orbits of $\mathrm{Ad}(u)$ and $\mathrm{ad}(x)$ coincide.

Lawther recently showed that the Jordan block sizes under the adjoint representation for unipotent elements and corresponding nilpotent element coincide. Furthermore, for minimal dimensional representations he has also shown that the Jordan block sizes coincide except for one case when $p = 5$ for the regular element in E_7. The partition for the unipotent class is (24, 22, 10) while the partition for the nilpotent element is $(23^2, 10)$.

In this paper, we are only considering orbits in $\mathcal{N}_1(\mathfrak{g})$ so this class will not come into play. Therefore, we can employ the tables given by Lawther in [23,24] which compute the partitions corresponding to unipotent classes under minimal dimensional representations. From [23, Table C] one can deduce that the orbit faithful condition is always satisfied for any given minimal dimensional representation. If $\rho : G \to \mathrm{SL}(V)$ is a minimal dimensional representation with $N = \dim V$ then $N = 27$ (resp. 56, 248, 26, 7) for E_6 (resp. E_7, E_8, F_4, G_2).

4.4.

The minimal dimensional representations along with Proposition 4.2 in conjunction with Lawther's tables can be used to carefully analyze what happens when orbit closures are intersected with $\mathfrak{g}$. This allows us to answer (1.1.3) for Φ of type G_2 or F_4.

Theorem 4.4. *Let G be an exceptional simple group where $\Phi = F_4$, or G_2 with p good. If $\mathcal{O}$ is an orbit in $\mathcal{N}_1(\mathfrak{g})$ then there exists a G-module M such that $\mathcal{V}_{\mathfrak{g}}(M) = \overline{\mathcal{O}}$.*

Proof. For $\Phi = G_2$, we see that the orbits are linearly ordered under $\preccurlyeq$. Therefore, by Proposition 4.2(a), for any orbit $\mathcal{O}$ for G_2, $\overline{\mathcal{O}}^{\lambda} \cap \mathfrak{g} = \overline{\mathcal{O}}$ for some partition λ of 7. From [7, Theorem 4.8], there exists $H^0_K(\sigma)$ for $\mathrm{SL}(V)$ such that $\mathcal{V}_{\mathfrak{sl}(V)}(H^0_K(\sigma)) = \overline{\mathcal{O}}^{\lambda}$. It follows by (2.1.5), that $\mathcal{V}_{\mathfrak{g}}(H^0_K(\sigma)) = \overline{\mathcal{O}}$.

For $\Phi = F_4$, all orbit closures can be realized as intersections with an orbit for $\mathfrak{sl}(\mathfrak{g})$ except for the orbit $\mathcal{O}(C_3)$ (when $p \neq 7$) and $\mathcal{O}(\widetilde{A}_2)$ (when $p \geqslant 5$) because

$$\overline{\mathcal{O}}^{(9,6^2,5)} \cap \mathfrak{g} = \overline{\mathcal{O}(C_3)} \cup \overline{\mathcal{O}(B_3)},$$
$$\overline{\mathcal{O}}^{(5,3^7)} \cap \mathfrak{g} = \overline{\mathcal{O}(\widetilde{A}_2)} \cup \overline{\mathcal{O}(A_2 + \widetilde{A}_1)}.$$

However, the orbits $\mathcal{O}(C_3)$ and $\mathcal{O}(\widetilde{A}_2)$ are Richardson so by Theorem 2.4 one can realize their orbit closures as a support varieties of (simple) tilting modules for G. □

4.5.

We begin our discussion for the exceptional cases of type E. For $\Phi = E_6$, one can realize all orbit closures as intersections of orbits for $\mathfrak{sl}(V)$ except for the orbit $\mathcal{O}(A_5)$. Checking this process again involves using Proposition 4.2 with Lawther's tables. From this analysis with the Hasse diagram for E_6, we have for $p \geqslant 5$;

$$\overline{\mathcal{O}}^{\lambda_1} \cap \mathfrak{g} = \overline{\mathcal{O}(A_5)} \cup \overline{\mathcal{O}(D_5(a_1))} \quad p \neq 7.$$

Also, note that for $p = 7$, one has

$$\overline{\mathcal{O}}^{\lambda_2} \cap \mathfrak{g} = \overline{\mathcal{O}(A_5)}.$$

The partition labels for λ_1 and λ_2 are given in the tables in the Appendix.

In the case when $p = 7$ the orbit closure of $\overline{\mathcal{O}(A_5)}$ can be realized as the support variety of a rational G-module. This shows that all orbit closures except possibly $\overline{\mathcal{O}(A_5)}$ when $p \neq 7$ can be realized as support varieties of G-modules.

4.6.

Let $\Phi = E_7$. By analyzing the Hasse diagram and applying Proposition 4.2 we find that there are six non-Richardson orbits whose closures cannot be realized as an intersection with the closure of an $\mathrm{SL}(V)$-orbit. These orbits are $\mathcal{O}(D_6)$, $\mathcal{O}(D_6(a_2))$, $\mathcal{O}(A_5')$, $\mathcal{O}(D_4 + A_1)$, $\mathcal{O}(A_5 + A_1)$, and $\mathcal{O}(4A_1)$. The decompositions are given below ($p \geqslant 5$). For an explicit description of μ_j, $j = 1, 2, \ldots, 5$, see the tables in the Appendix.

$$\begin{aligned}
&\overline{\mathcal{O}}^{\mu_1} \cap \mathfrak{g} = \overline{\mathcal{O}(D_6)} \cup \overline{\mathcal{O}(E_6(a_1))},\\
&\overline{\mathcal{O}}^{\mu_2} \cap \mathfrak{g} = \overline{\mathcal{O}(D_6(a_2))} \cup \overline{\mathcal{O}(E_6(a_3))},\\
&\overline{\mathcal{O}}^{\mu_3} \cap \mathfrak{g} = \overline{\mathcal{O}(A_5')} \cup \overline{\mathcal{O}(D_5(a_1) + A_1)} \quad p \neq 7,\\
&\overline{\mathcal{O}}^{\mu_4} \cap \mathfrak{g} = \overline{\mathcal{O}(D_4 + A_1)} \cup \overline{\mathcal{O}(A_4)},\\
&\overline{\mathcal{O}}^{\mu_5} \cap \mathfrak{g} = \overline{\mathcal{O}(A_5 + A_1)} \cup \overline{\mathcal{O}(D_5(a_1) + A_1)} \quad p \neq 7,\\
&\overline{\mathcal{O}}^{\mu_6} \cap \mathfrak{g} = \overline{\mathcal{O}(4A_1)} \cup \overline{\mathcal{O}(A_2)}.
\end{aligned}$$

Therefore, all the closures of orbits for E_7 with the exception of 6 possible orbits $\mathcal{O}(D_6)$, $\mathcal{O}(D_6(a_2))$, $\mathcal{O}(A_5')$, $\mathcal{O}(D_4 + A_1)$, $\mathcal{O}(A_5 + A_1)$, and $\mathcal{O}(4A_1)$ can be realized as the support variety of a G-module.

4.7.

Finally let $\Phi = E_8$. First, we determine the non-Richardson orbits whose closures can be realized by intersecting with the closure of an $\mathrm{SL}(V)$-orbit by using Proposition 4.2. All such orbits can be realized as support varieties of G-modules. There are 13 non-Richardson orbits which are not obtained in this fashion: $\mathcal{O}(E_7)$, $\mathcal{O}(D_7)$, $\mathcal{O}(E_7(a_2))$, $\mathcal{O}(E_6+A_1)$, $\mathcal{O}(D_6)$, $\mathcal{O}(E_7(a_4))$, $\mathcal{O}(D_5 + A_1)$, $\mathcal{O}(D_6(a_2))$, $\mathcal{O}(A_5 + A_1)$, $\mathcal{O}(A_5)$, $\mathcal{O}(D_5(a_1) + A_1)$, $\mathcal{O}(D_5(a_1))$, $\mathcal{O}(D_4+A_1)$. We list below the intersection of the corresponding $\mathrm{SL}(V)$-orbit with $\mathfrak{g}$ ($p \geqslant 7$). Again see the tables in the Appendix for the description of γ_j, $j = 1, 2, \ldots, 13$.

$$\begin{aligned}
&\overline{\mathcal{O}}^{\gamma_1} \cap \mathfrak{g} = \overline{\mathcal{O}(E_7)} \cup \overline{\mathcal{O}(E_8(a_4))},\\
&\overline{\mathcal{O}}^{\gamma_2} \cap \mathfrak{g} = \overline{\mathcal{O}(D_7)} \cup \overline{\mathcal{O}(E_8(b_5))} \quad p \neq 7, 17,\\
&\overline{\mathcal{O}}^{\gamma_3} \cap \mathfrak{g} = \overline{\mathcal{O}(E_7(a_2))} \cup \overline{\mathcal{O}(E_8(a_6))},\\
&\overline{\mathcal{O}}^{\gamma_4} \cap \mathfrak{g} = \overline{\mathcal{O}(E_6 + A_1)} \cup \overline{\mathcal{O}(D_7(a_1))},\\
&\overline{\mathcal{O}}^{\gamma_5} \cap \mathfrak{g} = \overline{\mathcal{O}(D_6)} \cup \overline{\mathcal{O}(D_7(a_2))},\\
&\overline{\mathcal{O}}^{\gamma_6} \cap \mathfrak{g} = \overline{\mathcal{O}(E_7(a_4))} \cup \overline{\mathcal{O}(A_6 + A_1)},\\
&\overline{\mathcal{O}}^{\gamma_7} \cap \mathfrak{g} = \overline{\mathcal{O}(D_5 + A_1)} \cup \overline{\mathcal{O}(E_8(a_7))},\\
&\overline{\mathcal{O}}^{\gamma_8} \cap \mathfrak{g} = \overline{\mathcal{O}(D_6(a_2))} \cup \overline{\mathcal{O}(E_6(a_3) + A_1)},\\
&\overline{\mathcal{O}}^{\gamma_9} \cap \mathfrak{g} = \overline{\mathcal{O}(A_5 + A_1)} \cup \overline{\mathcal{O}(D_5(a_1) + A_2)} \quad p \neq 7,
\end{aligned}$$

$$\overline{\mathcal{O}}^{\gamma_{10}} \cap \mathfrak{g} = \overline{\mathcal{O}(A_5)} \cup \overline{\mathcal{O}(D_5(a_1) + A_1)} \quad p \neq 7,$$
$$\overline{\mathcal{O}}^{\gamma_{11}} \cap \mathfrak{g} = \overline{\mathcal{O}(D_5(a_1) + A_1)} \cup \overline{\mathcal{O}(A_4 + A_2 + A_1)},$$
$$\overline{\mathcal{O}}^{\gamma_{12}} \cap \mathfrak{g} = \overline{\mathcal{O}(D_5(a_1))} \cup \overline{\mathcal{O}(2A_3)},$$
$$\overline{\mathcal{O}}^{\gamma_{13}} \cap \mathfrak{g} = \overline{\mathcal{O}(D_4 + A_1)} \cup \overline{\mathcal{O}(D_4(a_1) + A_2)} \cup \overline{\mathcal{O}(A_4)}.$$

We can realize three of these orbit closures using appropriate intersection techniques. From the Hasse diagrams, observe that

$$\overline{\mathcal{O}(E_7(a_4))} = \overline{\mathcal{O}(E_6(a_1))} \cap \overline{\mathcal{O}(D_5 + A_2)},$$
$$\overline{\mathcal{O}(D_5(a_1) + A_1)} = \overline{\mathcal{O}(E_6(a_3))} \cap \overline{\mathcal{O}(D_4 + A_2)}.$$

This shows that $\overline{\mathcal{O}(E_7(a_4))}$ and $\overline{\mathcal{O}(D_5(a_1) + A_1)}$ can be expressed as the intersection of the closures of Richardson orbits (using [16]). Therefore, by (2.1.2) and [7, Theorem 4.8], it follows that these two orbit closures can be realized as support varieties of G-modules.

Now consider the orbit $\mathcal{O}(A_5)$. From our list above, there exists a G-module M such that $\mathcal{V}_{\mathfrak{g}}(M) = \overline{\mathcal{O}(D_6(a_2))} \cup \overline{\mathcal{O}(E_6(a_3) + A_1)}$. Moreover, there exists a G-module N such that $\mathcal{V}_{\mathfrak{g}}(N) = \overline{\mathcal{O}(E_6(a_3))}$. By using the Hasse diagram one can verify that

$$\mathcal{V}_{\mathfrak{g}}(M \otimes N) = [\overline{\mathcal{O}(D_6(a_2))} \cup \overline{\mathcal{O}(E_6(a_3) + A_1)}] \cap [\overline{\mathcal{O}(E_6(a_3))}] = \overline{\mathcal{O}(A_5)}.$$

4.8.

The following theorem summarizes our findings for the exceptional groups E_6, E_7 and E_8 by combining the results in Sections 4.5–4.7 along with the explicit description of the restricted nullcone given in [7, Section 4.4]. Note that in some instances the orbits which cannot be realized lie outside the restricted nullcone (e.g. when $p = 5$) which allows us to exclude these cases.

Theorem 4.8. *Let G be an exceptional simple group where $\Phi = E_6, E_7, E_8$ with p good. If $\mathcal{O}$ is an orbit in $\mathcal{N}_1(\mathfrak{g})$ then there exists a G-module M such that $\mathcal{V}_{\mathfrak{g}}(M) = \overline{\mathcal{O}}$ in all cases with the (possible) exception of the following orbits.*

(i) $\Phi = E_6$ *and* $p \geqslant 11$: $\mathcal{O} = \mathcal{O}(A_5)$;
(ii) $\Phi = E_7$: $\mathcal{O} = \mathcal{O}(D_6)$ $p \geqslant 11$, $\mathcal{O}(D_6(a_2))$ $p \geqslant 7$, $\mathcal{O}(A_5')$ $p \geqslant 11$, $\mathcal{O}(D_4 + A_1)$ $p \geqslant 7$, $\mathcal{O}(A_5 + A_1)$ $p \geqslant 11$, $\mathcal{O}(4A_1)$ $p \geqslant 5$;
(iii) $\Phi = E_8$: $\mathcal{O}(E_7)$ $p \neq 7, 11, 13, 17$, $\mathcal{O}(D_7)$ $p \neq 7, 11, 17$, $\mathcal{O}(E_7(a_2))$ $p \neq 7, 11$, $\mathcal{O}(E_6 + A_1)$ $p \neq 7, 11$, $\mathcal{O}(D_6)$ $p \neq 7$, $\mathcal{O}(D_5 + A_1)$ $p \neq 7$, $\mathcal{O}(D_6(a_2))$, $\mathcal{O}(A_5 + A_1)$ $p \neq 7$, $\mathcal{O}(D_5(a_1))$, $\mathcal{O}(D_4 + A_1)$.

We remark that using the adjoint representation for $\Phi = E_7$, one can actually realize the orbit closure of $\mathcal{O}(4A_1)$ as the support variety of a G-module.

4.9.

Let $\rho : G \to \mathrm{SL}(V)$ and $N = \dim \mathfrak{g}$ be as in Section 4.3. The duality τ on $\mathrm{SL}(V)$-modules restricts to the transposed duality on G-modules. Therefore, for $i = 1, 2, \dots N - 1$, $H^0_K(\omega_i) \cong \Lambda^i(V) \cong \Lambda^i(V)^\tau = H^0_K(\omega_i)^\tau$. Also, $\Lambda^i(V)^* \cong \Lambda^{N-i}(V)$. From these facts, it follows that in order to show that $H^0_K(\omega_i)|_G$ has a good G-filtration for all i, it suffices to show that $H^0_K(\omega_i)|_G$ has a good G-filtration for all $i = 1, 2, \dots [(N+1)/2]$. Assuming this is true, one can deduce by Proposition 3.1, if M is a tilting module for $\mathrm{SL}(V)$ then $M|_G$ is a tilting G-module. When the prime is sufficiently large, one can provide an affirmative answer to (1.1.5) for the exceptional Lie algebras with the (possible) exceptions of the orbits listed in Theorem 4.8.

Theorem 4.9. *Let G be an exceptional simple algebraic group and let $\rho : G \to \mathrm{SL}(V)$ be a minimal dimensional representation of G. Assume the following conditions on the primes for ρ:*

(i) G_2: $p \geqslant 5$,
(ii) F_4: $p \geqslant 17$,
(iii) E_6: $p \geqslant 17$,
(iv) E_7: $p \geqslant 29$,
(v) E_8: $p \geqslant 97$.

Let $\mathcal{O}$ be an orbit in $\mathcal{N}_1(\mathfrak{g})$ which is not listed in Theorem 4.8. *Then there exists a tilting G-module M such that $\mathcal{V}_G(M) = \overline{\mathcal{O}}$.*

Proof. The G-module $\Lambda^i(V)$ is a summand of $V^{\otimes i}$ for $p > i$ [1, 4.1(5)] and thus has a good G-filtration. Therefore, for $p > [(N+1)/2]$, one can say that if M is a tilting module for $\mathrm{SL}(V)$ then $M|_G$ is a tilting G-module. Observe that the bound on the prime for E_8 is much smaller than $[(N+1)/2]$. The minimal representation for E_8 is the adjoint representation and it was shown that $\Lambda^i(V)$ is semisimple for $p > 3h - 3$ [13, Proposition 1.1].

Now by using the procedures given in Sections 4.4–4.7 along with Theorem 2.4 one can assume that M is a tilting module (by choosing an induced module with minimal weight). Also note that the tensor product of tilting modules is also a tilting module so the reductions in Section 4.9 are also valid for these primes. □

Acknowledgements

The first author would like to express his gratitude to the Department of Mathematics at Osaka City University and the Maths Institute at Oxford University for their hospitality during the writing of this paper. We thank Brian Boe, Masaharu Kaneda, Eric Sommers, and Donna Testerman for several useful discussions at various stages of this project, and RuthElizabeth Conine for technical assistance. The authors also acknowledge the helpful suggestions made by the referee.

Appendix A. Tables

A.10.

The following tables describe the partition labels in Sections 4.5–4.7 where p is a good prime.

Type E_6	Label	Prime	Partition
	λ_1	$5, \geqslant 11$	$(9, 6^2, 5, 1)$
	λ_2	7	$(7^2, 6^2, 1)$

Type E_7	Label	Prime	Partition
	μ_1	$5, 7, \geqslant 13$	$(16, 11^2, 10, 6, 1^2)$
		11	$(11^4, 10, 1^2)$
		13	$(13^2, 11^2, 6, 1^2)$
	μ_2	$5, \geqslant 11$	$(10, 8, 7^2, 5^4, 4)$
		7	$(7^6, 5^2, 4)$
	μ_3	$5, \geqslant 11$	$(9^2, 6^4, 5^2, 1^4)$
	μ_4	$5, \geqslant 11$	$(8, 7^4, 6, 2^5, 1^4)$
		7	$(7^6, 2^5, 1^4)$
	μ_5	$5, \geqslant 11$	$(10, 7^2, 6^3, 5^2, 4)$
	μ_6	$\geqslant 5$	$(4, 3^6, 2^{14}, 1^6)$

Type E_8	Label	Prime	Partition
	γ_1	7	$(35, 28^2, 21^4, 14^4, 7^2, 1^3)$
		$11, \geqslant 37$	$(35, 28^2, 27, 23, 19, 18^2, 15, 11, 10^2, 3, 1^3)$
		13	$(35, 28^2, 27, 23, 19, 18^2, 13^2, 10^2, 3, 1^3)$
		17	$(35, 28^2, 27, 23, 18^2, 17^2, 11, 10^2, 3, 1^3)$
		19	$(19^{11}, 18^2, 1^3)$
		23	$(23^9, 15, 10^2, 3, 1^3)$
		29	$(29^2, 28^2, 27, 19, 18^2, 15, 11, 10^2, 3, 1^3)$
		31	$(31^2, 28^2, 23, 19, 18^2, 15, 11, 10^2, 3, 1^3)$
	γ_2	11	$(23, 22^2, 19, 16^2, 13^3, 11^7, 4^2, 3, 1^3)$
		13	$(13^{17}, 12^2, 1^3)$

Type E_8	Label	Prime	Partition
		19	$(19^7, 13^3, 12^2, 11, 10^2, 7, 4^2, 3, 1^3)$
		$\geqslant 23$	$(23, 22^2, 19, 16^2, 15, 13^3, 12^2, 11, 10^2, 7, 4^2, 3, 1^3)$
	γ_3	$7, 11, \geqslant 23$	$(23, 19, 18^2, 17, 16^2, 15^2, 11^2, 10^2, 9, 8^2, 7, 4^2, 3^2, 1^3)$
		13	$(13^8, 4^2, 3, 1^3)$
		17	$(17^9, 15, 11, 10^2, 9, 8^2, 7, 4^2, 3^2, 1^3)$
		19	$(19^3, 18^2, 17, 16^2, 15, 11^2, 10^2, 9, 8^2, 7, 4^2, 3^2, 1^3)$
	γ_4	$7, 11, \geqslant 23$	$(23, 18^2, 17^3, 16^2, 15, 11, 10^2, 9^3, 8^2, 3^2, 2^4, 1^3)$
		13	$(13^{18}, 3, 2^4, 1^3)$
		17	$(17^9, 15, 10^2, 9^3, 8^2, 3^2, 2^4, 1^3)$
		19	$(19^2, 18^2, 17^3, 16^2, 11, 10^2, 9^3, 8^2, 3^2, 2^4, 1^3)$
	γ_5	$7, \geqslant 19$	$(19, 16^4, 15, 11^6, 10^4, 7, 6^4, 3, 1^{10})$
		11	$(11^{18}, 10^4, 1^{10})$
		13	$(13^{12}, 11^5, 6^4, 3, 1^{10})$
		17	$(17^2, 16^4, 11^6, 10^4, 7, 6^4, 3, 1^{10})$
	γ_6	$7, \geqslant 17$	$(15, 13, 12^2, 11^4, 10^4, 9^2, 8^2, 7^3, 6^2, 5^2, 4^4, 3^4, 2^2, 1^3)$
		11	$(11^{12}, 10^2, 9, 8^2, 7^2, 6^2, 5^2, 4^4, 3^4, 2^2, 1^3)$
		13	$(13^3, 12^2, 11^3, 10^4, 9^2, 8^2, 7^3, 6^2, 5^2, 4^4, 3^4, 2^2, 1^3)$
	γ_7	$7, \geqslant 17$	$(15, 12^2, 11^5, 10^4, 9^3, 8^2, 7, 6^2, 5^4, 4^2, 3^2, 2^6, 1^6)$
		11	$(11^{11}, 10^2, 9^3, 8^2, 6^2, 5^4, 4^2, 3^2, 2^6, 1^6)$
		13	$(13^2, 12^2, 11^4, 10^4, 9^3, 8^2, 7, 6^2, 5^4, 4^2, 3^2, 2^6, 1^6)$
	γ_8	7	$(7^{29}, 5^4, 4^4, 3, 1^6)$
		$\geqslant 11$	$(11^2, 10^4, 9, 8^4, 7^7, 6^4, 5^5, 4^8, 3^3, 1^6)$
	γ_9	$\geqslant 11$	$(11, 10^4, 9^3, 8^2, 7^5, 6^8, 5^7, 4^4, 3^2, 2^4, 1^6)$
	γ_{10}	$\geqslant 11$	$(11, 10^2, 9^7, 7, 6^{14}, 5^7, 4^2, 3, 1^{17})$
	γ_{11}	7	$(7^{28}, 4^2, 3^6, 2^{10}, 1^6)$
		$\geqslant 11$	$(11, 9^3, 8^6, 7^8, 6^6, 5^3, 4^2, 3^7, 2^{10}, 1^6)$
	γ_{12}	7	$(7^{28}, 3^7, 2^8, 1^{15})$
		$\geqslant 11$	$(11, 9, 8^8, 7^8, 6^8, 5, 3^8, 2^8, 1^{15})$
	γ_{13}	7	$(7^{28}, 3, 2^{14}, 1^{21})$
		$\geqslant 11$	$(11, 8^6, 7^{14}, 6^6, 3^2, 2^{14}, 1^{21})$

References

[1] H.H. Andersen, J.C. Jantzen, Cohomology of induced representations for algebraic groups, Math. Ann. 269 (1984) 487–525.

[3] R. Bezrukavnikov, Cohomology of tilting modules over quantum groups, and *t*-structures on derived categories of coherent sheaves (arXiv:math.RT/0403003).

[4] J.F. Carlson, The varieties and cohomology ring of a module, J. Algebra 85 (1983) 104–143.

[7] J.F. Carlson, Z. Lin, D.K. Nakano, B.J. Parshall, The restricted nullcone, Contemp. Math. 325 (2003) 51–75.

[8] R.W. Carter, Finite Groups of Lie Type, Wiley-Interscience, New York, 1985.

[9] D.H. Collingwood, W.M. McGovern, Nilpotent Orbits in Semisimple Lie Algebras, Van Nostrand Reinhold, New York, 1993.

[10] S. Donkin, Rational Representations of Algebraic Groups: Tensor Products and Filtrations, Springer-Verlag Lecture Notes, vol. 1140, Springer, Berlin, 1985.

[11] E.B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, Amer. Math. Soc. Transl. 6 (1957) 111–244.

[12] E.M. Friedlander, B.J. Parshall, Support varieties for restricted Lie algebras, Invent. Math. 86 (1986) 553–562.

[13] E.M. Friedlander, B.J. Parshall, Cohomology of Lie algebras and algebraic groups, Amer. J. Math. 108 (1986) 235–253.

[14] E.M. Friedlander, A.A. Suslin, Cohomology of finite group schemes over a field, Invent. Math. 127 (2) (1997) 209–270.

[15] W.H. Hesselink, Singularities in the nilpotent scheme of a classical group, Trans. Amer. Math. Soc. 222 (1976) 1–32.

[16] T. Hirai, On Richardson classes of unipotent elements in semisimple algebraic groups, Proc. Japan Acad. Ser. A 57 (1981) 367–372.

[17] J.E. Humphreys, Conjugacy classes in semisimple algebraic groups, Mathematical Surveys and Monographs, American Mathematical Society, vol. 43, 1995.

[18] J.E. Humphreys, Comparing modular representations of semisimple groups and their Lie algebras, Modular Interfaces, AMS/IP Stud. Adv. Math. (4) (1997) 69–80.

[19] J.C. Jantzen, Kohomologie von *p*-Lie-Algebren und nilpotente Elemente, Abh. Math. Sem. Univ. Hamburg 56 (1986) 191–219.

[20] J.C. Jantzen, Support varieties of Weyl modules, Bull. London Math. Soc. 19 (1987) 238–244.

[21] J.C. Jantzen, Representations of Algebraic Groups, second ed., American Mathematical Society, Providence RI, 2003.

[23] R. Lawther, Jordan block sizes of unipotent elements in exceptional algebraic groups, Comm. Algebra 23 (11) (1995) 4125–4156.

[24] R. Lawther, Correction to Jordan block sizes of unipotent elements in exceptional algebraic groups, Comm. Algebra 26 (8) (1998) 2709.

[25] O. Mathieu, Filtrations of *G*-modules, Ann. Sci. Ecole Norm. Sup. 23 (1990) 624–644.

[26] G.J. McNinch, Filtrations and positive characteristic Howe duality, Math. Z. 235 (2000) 651–685.

[27] G.J. McNinch, Abstract Jordan blocks, preprint, 2003.

[28] K. Mizuno, The conjugate classes of Chevalley groups of type E_6, J. Fac. Math. Sci. Tokyo 24 (1977) 525–563.

[29] K. Mizuno, The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8, Tokyo J. Math. 3 (1980) 391–461.

[30] D.K. Nakano, B.J. Parshall, D.C. Vella, Support varieties for algebraic groups, J. Reine Angew. Math. 547 (2002) 15–49.

[31] J. Pevtsova, Infinite dimensional modules for Frobenius kernels, J. Pure Appl. Algebra 173 (2002) 59–86.

[32] A.A. Premet, I.D. Suprunenko, The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights, Comm. Algebra 11 (1983) 1309–1342.

[33] T. Shoji, On the Green polynomials of a Chevalley group of type F_4, Comm. Algebra 10 (1982) 505–543.

[34] A.A. Suslin, E.M. Friedlander, C.P. Bendel, Infinitesimal one-parameter subgroups and cohomology, J. Appl. Math. Simulation 10 (1997) 693–728.